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Transition amplitudes for time-dependent linear harmonic oscillators

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Abstract

Following standard operator techniques, we obtain the transition amplitudes for a general time-dependent linear harmonic oscillator without using the Green function, the calculation of which is quite difficult. Our method provides a straightforward and manifest method for calculating these transition amplitudes.

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1. Introduction

Time-dependent harmonic oscillators have been considered by many authors [1–4]. From the physical point of view, Parker [5] applied the alpha and beta coefficients of the problem to the cosmological creation of particles in an expanding universe. Earlier, Kanai [2] had considered a special simple form of the time-dependent linear oscillator. However, his model was very strongly criticized by Brittin [6] and Senitzky [7], for various reasons.

Landovitz *et al*, rejecting the criticism, proceeded to calculate the Green function [8] for the general form of Kanai's [2] model and used it to calculate the corresponding transition amplitudes [9]. Their calculation of the Green function is quite tedious, making it difficult to comprehend.

In this paper, we present a new technique that avoids the Green function by using standard operators to calculate the transition amplitudes for the general simple model in a transparent manner, anticipating that this approach will be relevant to other physical problems, including Senitzky's [7] complex model of the dissipative quantum mechanical oscillator.

This paper is organized as follows. In section 2, we explain the problem and obtain the transformed operators $x_+(t)$, $p_+(t)$, in terms of the non-transformed time-independent operators x, p, and the coefficients in the 'transition matrix'. In section 3, we calculate the corresponding transformed creation and annihilation operators. In section 4, we derive the recursion relations satisfied by these transition amplitudes. These recursion relations are used in section 5 to calculate the transition amplitudes in terms of the initial one, which is then evaluated to complete the calculation.

2. The transformed operators $x_+(t)$, $p_+(t)$

The Hamiltonian for the time-dependent linear harmonic oscillator is given by

$$H(t) = f(t)\,\frac{p^2}{2m} + \frac{1}{2}g(t)\,m\omega^2 x^2 \tag{1}$$

where $f(0) = g(0) = 1$ gives the Hamiltonian for the time-independent situation and $f(t)$ and $g(t)$ are *real, continuous* functions to make the Hamiltonian Hermitian. For easy comparison, we have *generally* used the notation from [8]. Where we have used a different notation, we shall state it explicitly.

The wave functions at arbitrary time t are related to those at $t = 0$ through a time-dependent transformation $u(t)$ as

$$\psi(x,t) = u(t)\,\psi(x,0) \tag{2}$$

where

$$u(t)u^\dagger(t) = u^\dagger(t)u(t) = 1. \tag{3}$$

The Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x,t) = H(t)\,\psi(x,t) \tag{4}$$

and equation (2) give

$$\mathrm{i}\hbar\frac{\partial}{\partial t}u(x,t) = H(t)\,u(x,t) \tag{5}$$

which, for the time-independent Hamiltonian, results in the obviously unitary 'formal' solution:

$$u(t) = \mathrm{e}^{\frac{-\mathrm{i}}{\hbar}Ht}. \tag{6}$$

In general, however, equation (5) cannot be solved analytically.

We define the operator $O_\pm(t)$ corresponding to any operator $O(t)$ (which may have a manifest time dependence) by

$$O_+(t) = u^\dagger(t)O(t)u(t),\ O_-(t) = u(t)O(t)u^\dagger(t). \tag{7}$$

These operators satisfy the dynamic equations:

$$\frac{\partial}{\partial t}O_+(t) = \frac{1}{\mathrm{i}\hbar}[O_+(t),\ H_+(t)] + \left(\frac{\partial}{\partial t}O\right)_+ \tag{8a}$$

$$\frac{\partial}{\partial t}O_-(t) = \frac{1}{\mathrm{i}\hbar}\left[H(t),O_-(t)\right] + \left(\frac{\partial}{\partial t}O\right)_- \tag{8b}$$

which have a slight asymmetry. Note that the operator in (8*b*) in the commutator is $H(t)$, not $H_-(t)$ as may be expected if there were symmetry.

The operators $x_+(t)$, $p_+(t)$ are related to x, p through a linear transformation in terms of the 'transition matrix' $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ as

$$x_+(t) = u^\dagger(t)xu(t) = a(t)x + b(t)p \tag{9a}$$

$$p_+(t) = u^\dagger(t)pu(t) = c(t)x + d(t)p. \tag{9b}$$

The Hamiltonian equation (8*a*) implies

$$\begin{aligned} \dot a(t) &= \frac{f(t)}{m}c(t) \qquad & \dot b(t) &= \frac{f(t)}{m}d(t) \\ \dot c(t) &= m\omega^2 g(t)\,a(t) \qquad & \dot d(t) &= m\omega^2 g(t)\,b(t). \end{aligned} \tag{10}$$

Since, obviously

$$[x_+, p_+] = [x, p] = \mathrm{i}\hbar \tag{11}$$

we have

$$a(t)d(t) - b(t)c(t) = 1. \tag{12}$$

Solving equations (10), we obtain a, b, c, d as functions of t with

$$a(0) = d(0) = 1 \qquad b(0) = c(0) = 0. \tag{13}$$

These are then substituted in equation (9) to give the operators $x_+(t)$, $p_+(t)$.

Note that equations (10) imply

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\left(t\right)d\left(t\right) - b\left(t\right)c\left(t\right)\right) = 0$$

which again gives equation (12), using the conditions in equation (13). The transition matrix, thus, is unimodular for all t.

3. Calculation of the transformed creation and annihilation operators

The non-Hermitian creation and annihilation operators $A^\dagger$, A are related to the Hermitian operators x and p through

$$A^\dagger = \frac{-\mathrm{i}}{\sqrt{2m\hbar\omega}}(p + \mathrm{i}m\omega x) \tag{14a}$$

$$A = \frac{\mathrm{i}}{\sqrt{2m\hbar\omega}}(p - \mathrm{i}m\omega x). \tag{14b}$$

The above equation can be inverted to give

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(A + A^\dagger) \tag{15a}$$

$$p = -\mathrm{i}\sqrt{\frac{m\hbar\omega}{2}}\,(A + A^\dagger). \tag{15b}$$

The operators A, $A^\dagger$ satisfy the commutation relation

$$[A, A^\dagger] = 1. \tag{16}$$

In terms of the energy eigenstates of the Hamiltonian $H = H(0)$ given by

$$H|n\rangle = (n + \tfrac{1}{2})\hbar\omega|n\rangle \tag{17}$$

these operators have the matrix elements

$$\langle m|A|n\rangle = \sqrt{n}\delta_{mn-1} \qquad \langle m|A^\dagger|n\rangle = \sqrt{n+1}\delta_{mn+1}. \tag{18}$$

Next, we compute the transformed creation and annihilation operators $A_+^\dagger(t)$, $A_+(t)$ in terms of the elements $a(t)$, $b(t)$, $c(t)$, $d(t)$ of the transition matrix. Indeed

$$\begin{aligned}
A_+(t) &= u^\dagger(t)Au(t)\\
&= \frac{\mathrm{i}}{\sqrt{2m\hbar\omega}}(p_+ - \mathrm{i}m\omega x_+)\\
&= \frac{\mathrm{i}}{\sqrt{2m\hbar\omega}}[(c(t)x + d(t)p) - \mathrm{i}m\omega(a(t)x + b(t)p)]\\
&= \frac{\mathrm{i}}{\sqrt{2m\hbar\omega}}\left[(c(t) - \mathrm{i}m\omega a(t))\sqrt{\frac{\hbar}{2m\omega}}(A + A^\dagger)\right.
\end{aligned}$$

$$+(d(t) - \mathrm{i}m\omega b(t))(-\mathrm{i})\sqrt{\frac{m\hbar\omega}{2}}\,(A - A^\dagger)\Big]$$

$$= \alpha(t)A + \beta(t)A^\dagger \tag{19}$$

where

$$\alpha(t) = \frac{\mathrm{i}}{2m\omega}\left[c(t) - m^2\omega^2 b(t) - \mathrm{i}m\omega(a(t) + d(t))\right] \tag{20a}$$

$$\beta(t) = \frac{\mathrm{i}}{2m\omega}\left[c(t) + m^2\omega^2 b(t) - \mathrm{i}m\omega(a(t) - d(t))\right]. \tag{20b}$$

Similarly,

$$A_+^\dagger(t) = u^\dagger(t)A^\dagger u(t) = \beta^*(t)A + \alpha^*(t)A^\dagger \tag{21}$$

which can also be derived from equation (19) by taking Hermitian adjoints of the two sides.

In terms of the notations used in [8], α and β are related to λ, μ and σ in that work by

$$\alpha(t) = -\frac{\mathrm{i}}{2\sigma(t)}\lambda^*(t) \qquad \beta(t) = \frac{\mathrm{i}}{2\sigma(t)}\mu^*(t) \tag{22}$$

where

$$\sigma(t) = \frac{1}{m\omega b(t)} \tag{23a}$$

$$\lambda(t) = 1 - \mathrm{i}\sigma(t)(a(t) + d(t)) + \sigma^2(t)(1 - a(t)d(t)) \tag{23b}$$

$$\mu(t) = 1 + \mathrm{i}\sigma(t)(a(t) - d(t)) - \sigma^2(t)(1 - a(t)d(t)). \tag{23c}$$

Note that, using equations (13) in (20),

$$\alpha(0) = 1 \qquad \beta(0) = 0 \tag{24}$$

as expected from equations (19) and (21). The quantities $\sigma(t)$, $\lambda(t)$ and $\mu(t)$ have complicated $t = 0$ behaviour. This is the reason why we have used a different notation.

We may also check that

$$\begin{aligned}\left[A_+(t),\ A_+^\dagger(t)\right] &= \left[\alpha(t)A + \beta(t)A^\dagger,\ \beta^*(t)A + \alpha^*(t)A^\dagger\right] \\ &= |\alpha(t)|^2 - |\beta(t)|^2 \\ &= 1\end{aligned} \tag{25}$$

using equations (20). This result could have been guessed from another equation:

$$\left[A_+(t),\ A_+^\dagger(t)\right] = u^\dagger\left[A,\ A^\dagger\right]u = u^\dagger u = 1. \tag{26}$$

Just for completeness, we may also examine the operators

$$A_-(t) = u(t)Au^\dagger(t) \qquad A_-^\dagger(t) = u(t)Au^\dagger(t) \tag{27}$$

$$x_-(t) = d(t)x - b(t)p \qquad p_-(t) = -c(t)x + a(t)p. \tag{28}$$

Equations (28) have been written using the inverse of the transition matrix in equation (9) which we note is unimodular as expressed in equation (12). Thus, to obtain $A_-(t)$, $A_-^\dagger(t)$ from $A_+(t)$, $A_+^\dagger(t)$, we have to make the replacements

$$a(t) \leftrightarrow d(t) \qquad b(t) \leftrightarrow -b(t) \qquad c(t) \leftrightarrow -c(t) \tag{29}$$

which result in

$$\alpha(t) \to \frac{\mathrm{i}}{2m\omega}[-c(t) + m^2\omega^2 b(t) - \mathrm{i}m\omega(a(t) + d(t))] = \alpha^*(t) \tag{30a}$$

$$\beta(t) \to \frac{\mathrm{i}}{2m\omega}[-c(t) - m^2\omega^2 b(t) + \mathrm{i}m\omega(a(t) - d(t))] = -\beta(t). \tag{30b}$$

Thus

$$u(t)A(t)u^\dagger(t) = A_-(t) = \alpha^*(t)A(t) - \beta(t)A^\dagger(t) \tag{31a}$$

$$u(t)A^\dagger(t)u^\dagger(t) = A_-^\dagger(t) = -\beta^*(t)A(t) + \alpha(t)A^\dagger(t). \tag{31b}$$

4. Recursion relations for the transition amplitudes $\langle m|u(t)|n\rangle$

In this section, we derive the recursion relations satisfied by the transition amplitudes $a_{mn}(t) = \langle m|u(t)|n\rangle$ where the bras $\langle m|$ are eigenstates of $H(t)$ while the kets $|n\rangle$ are eigenstates of $H(0)$. Indeed,

$$\begin{aligned}\langle m|u(t)|n\rangle &= \frac{1}{\sqrt{n}}\langle m|u(t)A^\dagger(t)|n-1\rangle \\ &= \frac{1}{\sqrt{n}}\langle m|u(t)A^\dagger(t)u^\dagger(t)u(t)|n-1\rangle \\ &= \frac{1}{\sqrt{n}}\langle m|A_-^\dagger(t)u(t)|n-1\rangle \\ &= \frac{1}{\sqrt{n}}\langle m|(-\beta^*(t)A+\alpha(t)A^\dagger)u(t)|n-1\rangle\end{aligned}$$

using equation (31b)

$$\begin{aligned}&= \frac{1}{\sqrt{n}}[\langle m| - \beta^*(t)u(t)u^\dagger(t)Au(t)|n-1\rangle + \langle m|\alpha(t)A^\dagger(t)u(t)|n-1\rangle] \\ &= \frac{1}{\sqrt{n}}[\langle m| - \beta^*(t)u(t)(\alpha(t)A+\beta(t)A^\dagger)|n-1\rangle + \alpha(t)\sqrt{m}\langle m-1|u(t)|n-1\rangle] \\ &= \frac{1}{\sqrt{n}}\Big[-\alpha(t)\beta^*(t)\sqrt{n-1}\langle m|u(t)|n-2\rangle - |\beta(t)|^2\sqrt{n}\langle m|u(t)n\rangle \\ &\quad +\alpha(t)\sqrt{m}\langle m-1|u(t)|n-1\rangle\Big]\end{aligned}$$

which becomes

$$(1+|\beta(t)|^2)\langle m|u(t)|n\rangle = -\sqrt{\frac{n-1}{n}}\alpha(t)\beta^*(t)\langle m|u(t)|n-2\rangle + \sqrt{\frac{m}{n}}\alpha(t)\langle m-1|u(t)|n-1\rangle.$$

But, from equation (25), $1+|\beta(t)|^2 = |\alpha(t)|^2$. Thus, we have

$$\langle m|u(t)|n\rangle = -\frac{\beta^*(t)}{\alpha^*(t)}\sqrt{\frac{n-1}{n}}\langle m|u(t)|n-2\rangle + \frac{1}{\alpha^*(t)}\sqrt{\frac{m}{n}}\langle m-1|u(t)|n-1\rangle. \quad (32a)$$

A similar procedure results in another recursion relation

$$\langle m|u(t)|n\rangle = \frac{\beta(t)}{\alpha^*(t)}\sqrt{\frac{m-1}{m}}\langle m-2|u(t)|n\rangle + \frac{1}{\alpha^*(t)}\sqrt{\frac{n}{m}}\langle m-1|u(t)|n-1\rangle. \quad (32b)$$

5. Calculation of the transition amplitudes

We define the coefficients $B_{mn}(t)$ by means of the equation

$$a_{mn}(t) = \langle m|u(t)|n\rangle = \sqrt{m!n!}\left(\frac{\beta(t)}{\alpha^*(t)}\right)^{m/2}\left(\frac{\beta^*(t)}{\alpha^*(t)}\right)^{n/2}B_{mn}(t). \quad (33)$$

In terms of $B_{mn}(t)$, the recursion relations in equation (32) become

$$nB_{mn}(t) = -B_{mn-2}(t) + \frac{1}{|\beta|}B_{m-1n-1}(t) \quad (34)$$

and

$$mB_{mn}(t) = B_{m-2n}(t) + \frac{1}{|\beta|}B_{m-1n-1}(t). \quad (35)$$

The above recursion relations are sufficient to determine all $B_{mn}(t)$ and hence the transition amplitudes, up to a constant. For this purpose, we define a generating function

$$G(x, y, t) = \sum_{m,n=0}^{\infty} B_{mn}(t) x^m y^n. \tag{36}$$

To derive the partial differential equations satisfied by $G(x, y, t)$, we multiply equations (34) and (35) with $x^m y^n$ and add from 0 to ∞ for both m and n. We shall also assume that any $B_{mn}(t)$ with any of the indices m or n negative gives $B_{mn}(t) = 0$. We also note that

$$x\frac{\partial}{\partial x} G(x, y, t) = \sum_{m,n=0}^{\infty} m B_{mn}(t) x^m y^n$$

$$y\frac{\partial}{\partial x} G(x, y, t) = \sum_{m,n=0}^{\infty} m B_{mn}(t) x^m y^n$$

$$xyG(x, y, t) = \sum_{m,n=0}^{\infty} B_{mn}(t) x^{m+1} y^{n+1} = \sum_{m,n=0}^{\infty} B_{m-1n-1}(t) x^m y^n$$

$$x^2 G(x, y, t) = \sum_{m,n=0}^{\infty} B_{mn}(t) x^{m+2} y^n = \sum_{m,n=0}^{\infty} B_{m-2n}(t) x^m y^n$$

$$y^2 G(x, y, t) = \sum_{m,n=0}^{\infty} B_{mn}(t) x^m y^{n+2} = \sum_{m,n=0}^{\infty} B_{mn-2}(t) x^m y^n.$$

Then, from equations (34) and (35), we derive the partial differential equations

$$\frac{\partial}{\partial y} G(x, y, t) = \left(-y + \frac{1}{|\beta(x)|} x\right) G(x, y, t) \tag{37a}$$

$$\frac{\partial}{\partial x} G(x, y, t) = \left(x + \frac{1}{|\beta(x)|} y\right) G(x, y, t) \tag{37b}$$

which can be trivially solved to obtain

$$G(x, y, t) = G(0, 0, t) \mathrm{e}^{(x^2 - y^2)/2 + (1/|\beta|)xy}. \tag{38}$$

The t-dependent function $G(0, 0, t)$ is related to $a_{00}(t) = \langle 0|u(t)|0\rangle$. Indeed,

$$G(0, 0, t) = B_{00}(t) = a_{00}(t) = \langle 0|u(t)|0\rangle$$

and thus

$$G(x, y, t) = a_{00}(t) \mathrm{e}^{(x^2 - y^2)/2 + (1/|\beta|)xy} \tag{39}$$

$$= a_{00}(t) \sum_{i,j,k=0}^{\infty} \frac{(\frac{1}{2})^{i+j} x^{2i+k} y^{2j+k} (-1)^j}{|\beta(t)|^k i! j! k!}. \tag{40}$$

We now compare the coefficients of $x^m y^n$ in equations (36) and (40) by taking $m = 2i + k$, $n = 2j + k$, which fix i and j as $\frac{m-k}{2}$, $\frac{n-k}{2}$. Thus

$$B_{mn}(t) = a_{00}(t) \sum_k \frac{(\frac{1}{2})^{[(m+n)/2]-k} (-1)^{(n-k)/2}}{|\beta(t)|^k k! (\frac{m-k}{2})! (\frac{n-k}{2})!}. \tag{41}$$

Substituting for $B_{mn}(t)$ from equation (41) in (33), we arrive at

$$a_{mn}(t) = \langle m|u(t)|n\rangle = a_{00}(t) \frac{\sqrt{m!n!}}{(2\alpha^*)^{(m+n)/2}} \sum_k (-1)^{(n-k)/2} \frac{(\beta(t))^{(m-k)/2} (\beta^*(t))^{(n-k)/2}}{k! (\frac{m-k}{2})! (\frac{n-k}{2})!} \tag{42}$$

which gives the transition amplitudes $a_{mn}(t)$ in terms of an m-, n-independent, t-dependent function $a_{00}(t)$. *Note that the k summation starts from 0 or 1, depending upon whether m and n are both even or both odd, and goes up to* $\min(m, n)$ *in steps of 2*. Also it is obvious from the above that $a_{mn}(t) = 0$ if $|m - n| =$ odd integer.

To evaluate $a_{00}(t) = \langle 0|u(t)|0\rangle$, we put $m = 0$ in equation (32a) to find

$$\langle 0|u(t)|2n\rangle = -\sqrt{\frac{2n-1}{2n}}\frac{\beta^*(t)}{\alpha^*(t)}\langle 0|u(t)|2n-2\rangle \tag{43}$$

from which we obtain

$$\begin{aligned}\langle 0|u(t)|2n\rangle &= (-1)^n\sqrt{\frac{(2n-1)(2n-3)\ldots 1}{2n(2n-2)\ldots 2}}\left(\frac{\beta^*(t)}{\alpha^*(t)}\right)^2\langle 0|u(t)|0\rangle \\ &= (-1)^n\frac{\sqrt{(2n)!}}{2^n n!}\left(\frac{\beta^*(t)}{\alpha^*(t)}\right)^n\langle 0|u(t)|0\rangle.\end{aligned} \tag{44}$$

But

$$\begin{aligned}\langle 0|0\rangle = 1 &= \sum_{n=0}^{\infty}\langle 0|u(t)|2n\rangle\langle 2n|u^\dagger(t)|0\rangle = \sum_{n=0}^{\infty}|\langle 0|u(t)|2n\rangle|^2 \\ &= \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!n!}\left|\left(\frac{\beta^*(t)}{\alpha^*(t)}\right)\right|^{2n}|\langle 0|u(t)|0\rangle|^2.\end{aligned} \tag{45}$$

The summation over 'n' can be performed by using the duplication for the gamma function in the form

$$\begin{aligned}&(2n)! = \Gamma(2n+1) = \frac{n!\Gamma(n+\frac{1}{2})2^{2n}}{\sqrt{\pi}} \\ &\therefore 1 = \sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}\left|\frac{\beta^*(t)}{\alpha^*(t)}\right|^{2n}|\langle 0|u(t)|0\rangle|^2 \\ &\quad = |\langle 0|u(t)|0\rangle|^2\left(1-\left|\frac{\beta^*(t)}{\alpha^*(t)}\right|^2\right)^{-1/2} = |\langle 0|u(t)|0\rangle|^2|\alpha(t)| \\ &\therefore |\langle 0|u(t)|0\rangle| = \frac{1}{\sqrt{|\alpha(t)|}}.\end{aligned} \tag{46}$$

We choose the phase of $\langle 0|u(t)|0\rangle$ such that

$$\langle 0|u(t)|0\rangle = \frac{1}{\sqrt{|\alpha(t)|}} \tag{47}$$

which agrees with $\langle 0|0\rangle = 1$ when $t = 0$, in which case $u(0) = 1$ and $\alpha(0) = 1$, as given in equation (24).

Finally, then,

$$\begin{aligned}a_{mn}(t) &= \langle m|u(t)|n\rangle \\ &= \begin{cases}\sqrt{\dfrac{m!n!}{|\alpha(t)|}}\dfrac{1}{(2\alpha^*(t))^{(m+n)/2}}\displaystyle\sum_k\frac{(-1)^{(n-k)/2}2^k(\beta(t))^{(m-k)/2}(\beta^*(t))^{(n-k)/2}}{k!(\frac{m-k}{2})!(\frac{n-k}{2})!} \\ \qquad \text{when} \quad |m-n| = \text{even} \\ 0 \qquad \text{when} \quad |m-n| = \text{odd}.\end{cases}\end{aligned} \tag{48}$$

As in equation (42), the summation over k in equation (48) starts from 0 or 1, depending on whether m and n are both even or both odd, and goes to $\min(m, n)$ in steps of 2.

When $t \to 0, \alpha(t) \to 1, \beta(t) \to 0$. Thus, the only k value that survives in the summation is where $\frac{m-k}{2} = 0 = \frac{n-k}{2}$. In other words, we require $m = n = k$ and we find that

$$a_{mn}(0) = \langle m|u(0)|n\rangle = \langle m|n\rangle = \delta_{mn}$$

as expected from the orthonormality of the eigenstates of the time-independent Hamiltonian $H = H(0)$.

We note that, in our notation, this limit is obtainable trivially. With the notation in [8], some analysis is needed to arrive at this limit.

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